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THE ISAACS PROBLEM OF MOVING AROUND AN ISLAND*

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Two cutters, players, sail on the "sea", a fixed plane. A circular island of unit radius has its center at the origin of a fixed coordinate system. Outside the island the velocities of cutters are arbitrary as to direction and limited in modulo. At the "island" boundary (in coastal waters) cutter velocities are directed either out to "sea" or along a tangent ot the /island/ boundary. If the cutter is on the island its velocity is zero. The first player (fast cutter) minimizes the payoff, while the second player (slow cutter) maximizes it. In the first game the payoff is the distance between cutters at a fixed instant of time. In the second game the payoff is the time of convergence to a given distance. The difficulty of solving these problems which involve moving around an island was noted by Isaacs /1/. Both problems are solved in this paper.

1. The two-dimensional vectors $z_i = (z_{i,1}^\circ, z_{i,2}^\circ), \nu_i = (\mu_i, l_i)$ with $i = 1, 2$ are defined in the stationary system of coordinates X_1°, X_2° in the plane P. The number τ and vectors z_i, ν_i constitute vectors $x_{(i)} = (z_i, \nu_i, \tau)$. The two-dimensional controls $u_i = (u_{i,1}^\circ, u_{i,2}^\circ)$ with $i = 1, 2$ represent the velocities of players, and vectors $x_{(i)}$ enable the construction of vectors $x = (x_{(1)}, x_{(2)})$, $u = (u_1, u_2)$ and the equations of motion which are of the form

$$z_i^{\cdot} = u_i, \ \nu_i^{\cdot} = 0, \ \tau^{\cdot} + 1 = 0, \ i = 1, 2$$

The control $u \in \zeta(x) = \zeta_1(x) \times \zeta_2(x)$, where the sets ζ_i are of the form

$$\zeta_i(x) = \{u_i \mid |u_i| \leqslant \mu_i\}$$

when $|z_i| - 1 > 0$ (in "open sea"),

$$\zeta_i(x) = \{u_i \mid |u_i| \leqslant \mu_i, \ z_i u_i \geqslant 0\}$$

when $|z_i| - 1 = 0$ (in "coastal waters"),

$$\zeta_i(x) = \{u_i \mid |u_i| = 0\}$$

when $|z_i| - 1 < 0$ (on the "island").

Vector $x \in X$, where the set X is defined by the relations

$$X = \{x \mid (|z_i| - 1 \geqslant 0, \ i = 1, 2), \ \mu_1 > \mu_2 > 0, \ l_1 = l > 0, \ l_2 = \varepsilon > 0\}$$

We denote

$$r_1 = z_2 - z_1, \ r = |r_1|, \ n(x) = r - l$$
$$X_1^\circ = \{x \mid \tau \geqslant 0\} \cap X, \ X_2^\circ = \{x \mid n(x) \geqslant 0\}, \ z = (x_1, x) \in X \times X$$

Let us consider function $u_\xi(z) = (u_{\xi,1}, u_{\xi,2})$ and sets $\xi_i^\circ(x_1) \subset X_1$ defined by the relations

$$u_\xi(z) \subset \zeta(x), \ u_\xi(z) = \lim u_\xi(x_1, x_2) \ \text{as} \ x_2 \to x, \ x_2 \in X$$
$$\xi_i(x_1) \supset \{x \mid |x - x_1| - \varepsilon\} \cap X = \alpha_\varepsilon(x_1)$$

We combine functions $u_{\xi,i}(z)$ in the set $v_{\xi,i}$ and examine the sets

$$v_i = \{u_{\xi,i}(z), \xi_i(x_1)\}, v_{1,i} = \{x_1, v_1\}, w_{1,i} = \{x_1, u_\xi(z), \xi_i(x_1)\}$$

The motion $x_v(t)$ $(x_v(0) = x_1, t_1 = 0)$ of set $w_{1,i}$ is absolutely continuous, and the sequence $t_j, j = 1, 2, \ldots$ defined for $x_j = x_v(t_j)$ by the equality

$$t_{j+1} = \inf\{t \mid (t > t_j, x_v(t) \overline{\in} \xi_i(x_j)) \vee (t > t_j + 1)\}$$

is such that $t_j \to \infty$ as $j \to \infty$.

Function $l_i(x_1, x) = l(x, u_\xi(z))$ conforms to the equation $x_v^{\cdot}(t) = l_i(x_j, x_v(t))$ for almost all $t \in [t_j, t_{j+1}]$. Motions $x_v(t)$ and sets

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$$\sigma_\xi(v_{1,i}) = (\bigcup x_v(t) \text{ for } u_{\xi,j} \in v_{\xi,j} \text{ as } i \neq j)\, v_{t,i} = \{x_v(t), w_{1,i}\}$$

exist for all $w_{1,i}$.

We specify two functions

$$h_1(v_{t,i}) = r(x_v(\tau)),\ h_2 = \inf \theta_2(v_{t,i})$$
$$\theta_2(v_{t,i}) = \{t \mid t \geqslant 0,\ x_v(t) \in \{x \mid n(x) \geqslant 0\}\},\quad \theta_2 = \varnothing$$
$$h_2 = \infty$$

and calculate the series of functions

$$h_{i,j}(v_{1,i}) = (-1)^{i+1} \sup((-1)^{i+1} h_j(v_{1,t}) \text{ for } x_v(t) \subset \sigma_\xi(v_{1,i}))$$
$$r_{i,j}(x_1) = (-1)^{i+1} \inf((-1)^{i+1} h_{i,j}(v_{1,i}) \text{ for } v_i \in V_1, \varepsilon > 0)$$
$$V_{0,i,j} = \{v_i \mid \lim(h_{ij}(v_{1,i}) - r_{i,j}(x_1)) = 0 \text{ as } \varepsilon \to 0\} \supset v_{0,i,j}(z)$$

Function $r_{i,j}(x_1)$ is the value of the game of the i-th player in a game of number j, and $v_{0,i,j}(z)$ is the best strategy.

2. We introduce vectors $p_i = (p_{z,i}, p_{v,i}, p_r)$ whose structure is that of vector $x_{(i)}$. Sets

$$\varphi_i = \{p_i \mid |p_{z,i}| > 0\},\ \varphi = \varphi_1 \times \varphi_2,\ \gamma = \varphi \times X$$

contain vectors p_i, $p = (p_1, p_2)$, $y = (p, x)$. Vector $y^\circ = (-x, p)$.

We introduce the following notations: W_α, $W_{\alpha,2}$ for sets, and $W_{\alpha,1}$ for a vector (scalar). We shall use the letters $W, v, w, \xi, \eta, \zeta, \alpha, \gamma, \delta, \sigma, \theta, \varphi$ for denoting sets. We denote

$$a_{z,i}(y) = z_i,\ a_x(y) = x,\ a_\tau(x) = a_\tau(y) = \tau$$
$$a_{p,z,i}(y) = p_{z,i},\ a_{p,\mu,i}(y) = \lambda_i$$

The set $W_\alpha = \{W_{\alpha,1}, W_{\alpha,2}\}$ consists of vector (scalar) W_α and of set $W_{\alpha,2}$. Let $W_{\alpha,2}(y) \in \zeta$. Then

$$a_i^{\,i}(W_{\alpha,2}) = \{u_i \mid u \subset W_{\alpha,2}\} = a_i^{\,i}(W_{\alpha,2}, y, u_j)$$
$$a_i^{\,j}(W_{\alpha,2}) = \{u_j \mid a_i^{\,i}(W_{\alpha,2}, y, u_j) \neq \varnothing\}$$

We construct the operators

$$f_i^\circ(W_\alpha) = W_{i,\alpha} = \{W_{i,\alpha,1}, a_1^{\,i}(W_{\alpha,1}, \ldots) \times a_2^{\,i}(W_{i,\alpha,1}, \ldots)\}$$

for $W_{\alpha,1}(y, u) \in R^1$.

The space R^k is of dimension k. We construct operator $W_{1,\alpha}$ which is the operator of minimum for $i = 1$ (maximum for $i = 2$) using formulas

$$W_{i,\alpha,1} = (-1)^{i+1} \inf\{(-1)^{i+1} W_{\alpha,1}(y,u) \text{ for } u_i \in a_i^{\,i}(W_{\alpha,2}, y, u_j)\}$$
$$a_i^{\,i}(W_{i,\alpha,2}, y, u_j) = \{u_i \mid W_{i,\alpha,1} - W_{\alpha,1}(y,u) = 0,\quad u \in W_{\alpha,2}(y)\}$$
$$a_i^{\,j}(W_{i,\alpha,2}, y) = a_i^{\,j}(W_{\alpha,2}, y)$$

The construction is similar when $W_{\alpha,1} = c(x, p)$, $W_{\alpha,2} \subset \varphi$. If $W_{\alpha,1} = c(y, t)$, $W_{\alpha,2} \subset \{t\}$ then $f_{i,t}(W_\alpha) = f_i^\circ(W_\alpha)$. The minimax operators for $i = 1$ and maxmin operators when $i = 2$ are of the form $f_i(W_\alpha) = f_i^\circ(f_j^\circ(W_\alpha))$. Note that $c(y) \in R^1$, $h = pl(x, u)$.

Let us calculate

$$f_u(c(y)) = \lim(t^{-1}(c(y + (\partial h(y, u)/\partial y)\, t) - c(y)))$$

as $t \to 0$.

We set $c_t(y) \subset R^1$, $y_\alpha(y, t) \subset \gamma$, $\gamma_\alpha \subset \gamma$ and calculate for the set $v_\alpha = \{c_t(y), y_\alpha(y, t), \gamma_\alpha\}$ the boundary operator

$$W_a(v_\alpha) = \{t_a(v_\alpha), \tau_a(v_\alpha), y_a(v_\alpha), \theta_a(v_\alpha)\}$$
$$\theta_b = \{t \mid t \geqslant 0,\ y_\alpha(y,t) \in \gamma_\alpha\},\ t_a(v_\alpha) = \inf \theta_b(v_\alpha)$$
$$y_a = y_\alpha(y, t_a(v_\alpha)),\ \tau_a = c_t(y_a(v_\alpha)),\ \theta_a = \{t \mid t \in [0, t_a(v_\alpha)]\}$$
$$\varphi_a(x, \gamma_\alpha) = \{p \mid y \in \gamma_\alpha\}$$

We construct the sets

$$f_{i,\alpha}(v_\alpha) = f_i(\tau_a(y),\ \varphi_a(x, \gamma_\alpha))$$
$$\zeta_{i,\beta}(c(x), v_\alpha) = f_{i,t}(c(y_\alpha(y,t)), \theta_a(v_\alpha))$$
$$t_\lambda(c(y), v_\alpha) = \sup \zeta_{2,\beta,2}(y)$$
$$f_{i,b} = f_{2,a}(\zeta_{i,\beta,1}(\tau_a(y), v_\alpha), \theta_a(v_\alpha))$$
$$c_\xi(v_\alpha) = f_{2,a,1}(v_\alpha)$$

Functions

$$v_{i,\theta} = \{0, y_b(y, t), \gamma_{i,\theta}\},\ \partial y_b(y, t)/\partial t = l_h(y_b(y, t))$$

and sets

$$l_h = \{\partial h/\partial y^\circ \mid u \subset f_{2,1}(h,\ \zeta)\},\ \gamma_{i,\theta} = \gamma_{i,1} \cup \gamma_{i,2} \cup \gamma_{i,3}$$

correspond to sets $\alpha \subset X,\ \gamma^\circ(\alpha) = \{y \mid x \in \alpha\} \cap \gamma$.

We shall use the notation

$$\gamma_{i,1} = \gamma \cap \{x \mid |z_i| - 1 > 0\},\ \gamma_{i,2} = \gamma \cap \{x \mid |z_i| - 1 = 0$$
$$(-1)^i p_{z,i} z_i > 0\}$$
$$\gamma_{i,} \ = \gamma \cap \{x \mid \tau > 0\}$$

and determine the operators

$$t_{i,\theta}(x) = t_a(v_{i,\theta}),\ t_\theta(x) = \min\ (t_{1,\theta}(x),\ t_{2,\theta}(x))$$
$$\alpha_1 = \{x \mid t_\theta(x) \geqslant \tau\},\ \alpha_{1,i}\ (x) = \{x \mid t_{i,\theta}(x) = t_\theta(x) < \tau\}$$
$$f^\circ(c(x)) = \max\ (0,\ c(x))$$
$$x_{1,a}(x,\ t) = a_x(y_b(y,\ t)) \text{ for } y \in \gamma_{1,\theta} \cup \gamma_{2,\theta}$$

As the first step we determine $c_2 = r - (\mu_1 - \mu_2)\tau$

$$a_{1,\xi}(x) = a_{1,\xi}(y) = f^\circ(c_2(x)),\ \gamma_0(\alpha) = \gamma - \gamma^\circ(\alpha)$$
$$W_{1,\beta} = \{a_{1,\xi}(x),\ y_b(y,\ t),\ \gamma_0(\alpha_1)\},\ a_{2,\xi}(x) = c_\xi(W_{1,\beta})$$
$$v_{1,\beta} = \{-\varphi^\circ + \pi,\ y_b(y,\ t),\ \{\alpha_1 \times f_{1,b,1}(W_{1,\beta})\}\}$$

Note that by construction function $a_{1,\xi}(x) = a_{2,\xi}(x)$ for $x \subset \alpha_1$. We specify the sets

$$\alpha_\varphi = \{x \mid \varphi^\circ(x) \subset [0,\ \pi],\ x \subset \alpha_1,\ \tau \geqslant 0\}$$
$$x_{2,\xi} = \{a_x(y_b(y,t)) \mid p \subset f_{1,a,1}(W_{1,\beta})\}$$
$$v_\varphi = \{0,\ x_{2,\xi}(x,t)\,\alpha_\varphi\},\ t_{2,\varphi} = t_a(v_\varphi),\quad \varphi^\circ = \arccos(z_1 z_2 / |z_1| \cdot |z_2|) + k\pi$$

Let us define function $x_{2,\xi}(x,\ t)$ for $x \in \alpha_\varphi,\ t \leqslant t_{2,\varphi};\ \lambda(z_2,\ z_1) = \varphi^\circ(x),\ a_{2,\xi}(x)$.

The angle φ° is read counterclockwise from vector z_1 to vector z_2.

We specify the set

$$\xi_\lambda(x) = \{c_\lambda \mid c_\lambda \subset R^2,\ |c| - 1 = 0,\ \lambda(c_\lambda,\ z_1) \subset [0,\ \pi]\}$$

and write the sequence of functions

$$\alpha_{2,i}(x) = \alpha_{2,i}(z_i, x) = \sqrt{z_i^2 - 1} - \mu_i \tau$$
$$\alpha_{1,i}(z_i, x) = |z_i| - 1 - \mu_i \tau,\quad q_i(z_i, x) = \operatorname{arc\,tg} \sqrt{z_i^2 + 1}$$
$$q(x) = \varphi^\circ(x) + q_2(x) - q_1(x),\ \alpha_{3,i}(z_i,\ x) = q(x) - \alpha_{2,i}(x)$$

which enable us to define the sets $C_i = \xi_\lambda \cap \xi_{i,\lambda}$ and $\delta_{j,i} = \alpha_{1,i} \cap \mathring{\delta}_{j,i} \cap \alpha_\varphi$. When

$$\mathring{\delta}_{1,i} = \{x \mid \alpha_{2,i}(x) \geqslant 0\},\quad \mathring{\delta}_{2,i} = \{x \mid \alpha_{2,i}(x) < 0,\quad \alpha_{3,i}(x) \geqslant 0\}$$
$$\mathring{\delta}_{3,i} = \{x \mid \alpha_{3,i}(x) \leqslant 0\}$$

we have

$$C_i(x) = \xi_\lambda \cap \{c_\lambda \mid \alpha_{1,i}(c_\lambda, x) = 0\},\ C_j(x) = \xi_\lambda \cap \{c_\lambda \mid \alpha_{1,j}(c_\lambda, x) = 0\}$$

and for $x \in \delta_{1,k},\ i,\ j,\ k = 1,\ 2$

$$C_i(x) = \xi_\lambda \cap \{c_\lambda \mid \lambda\ (c,\ z_1) = q_i(x)\} \quad \text{for} \quad x \in \delta_{2,i} \cup \delta_{3,i};$$
$$i = 1,\ 2$$
$$C_i(x) = \xi_\lambda \cap \{c_\lambda \mid \lambda\ (c,\ z_1) = q_j(x) + \alpha_{2,j}(x)\} \quad \text{for} \quad x \in \delta_{3,j},$$
$$i \neq j$$
$$a_{2,\xi} = f^\circ(c_{2,\xi}(x))$$

The imbeddings $c_i(x) \in C_i(x) \subset R^1$ completely determine function $c_{2,\xi}(x)$.

The set R^1 is the join of one-element sets, and function $c_{2,\xi}$ is determined by the relations

$$c_{2,\xi} = |z_j - c_j(x)| - \mu_j \tau \quad \text{for } x \in \delta_{1,i} \cup \delta_{2,i},\quad i \neq j = 1, 2$$
$$c_{2,\xi} = |z_1 - c_1(x)| + |z_2 - c_2(x)| + q(x) - (\mu_1 - \mu_2)\tau$$
$$\text{for } x \in \delta_{3,1} \cup \delta_{3,2}$$

Let us define vector $p_{2,a} \subset \varphi_a(x)$ for $x \in \alpha\varphi$:

$$\varphi_a(x) = \{p \mid a_{p,z,j}(p) = (z_j - c_j(x))(-1)^{i+j-1}\}$$
$$\alpha_\varphi{}^\circ = \alpha_\varphi \setminus \{x \mid \varphi = 0, \pi\}$$

which we extend together with vector $y_{2,a} \subset X \times \varphi_a(x) = \gamma_{2,a}$ in continuity to the set

$$\alpha_{1,\varphi} = \{x \mid \varphi^\circ = \pi, m_\beta(x) < 0\}, \quad m_\beta = \zeta_{1,\beta,1}(\pi - \varphi^\circ, v_{1,\beta})$$

Let us now construct the equivocal surface δ_φ. For this we calculate

$$\beta_\xi(x, u) = f_u(a_{2,\xi}(x)), \ \beta_2(x) = f_2(\beta_\xi(x, u), \zeta(x))$$

We denote

$$h_\lambda = (1 - \cos q_1(x))\ \mu_1 - (1 - \cos q_2(x))\ \mu_2$$

and write down the sets

$$\xi_{1,\varphi} = \{x \mid x \in \alpha_{1,\varphi}, \ \beta_2(x) < 0\} = \xi_{2,\varphi} = \{x \mid \varphi = \pi,$$
$$\beta_2(x) = h_\lambda(x) < 0\}$$
$$\xi_\varphi = \xi_{1,\varphi} \cap \{x \mid h_\lambda(x) = 0\}; \ X_\varphi = X \times \varphi_a(x) \ \text{for} \ x \in \xi_\varphi$$

We construct the sequence of sets

$$\lambda_2 = \rho_{\mu,2}, \ h_2 = h(y, u) + \lambda_2\beta_\xi(x, u), \ W_h = f_1(h_2, \zeta)$$
$$\theta_\beta = \{y_1 \mid a_{\lambda,2}(y_1) = \lambda_{2,1}, \ a_0(y_1) = a_0(y), \ W_{h,2}(y_1) \subset \{u \mid \beta_\xi(x, u) = 0\}\}$$

The equality $a_0(y_1) = a_0(y)$ which follows equality $a_{\lambda,2}(y_1) = \lambda_{2,1}$, means that, generally $\lambda_{2,1} \neq \lambda_2$ and the remaining components of vectors y_1, y are the same. Let $\theta_\beta(y) = y_\beta(y) \subset R^1$. The motion $y_\rho(y, t)$ corresponds to relations

$$y_\rho(y, 0) = y_\beta(y) \ \text{for} \ y \subset \gamma_\varphi; \ \partial y_\rho/\partial t = l_\rho(y_\rho)$$
$$l_\rho = \{\partial h_2(y, u)/\partial y^\circ \mid u \in W_{h,2}(y)\}; \ y_\rho(y, t) \in X \times \varphi_a(x)$$
$$\text{when} \ t > 0$$

These conditions separate the unique motion

$$x_\rho(x, t) = a_x(y_\rho(y, t)) \ \text{for} \ y \in \gamma_\varphi$$

The sets

$$\xi_{1,\rho} = \{x \mid |z_2| - 1 > 0\}, \ W_{1,\rho} = \{0, \ x_\rho(x, -t), \ \xi_{1,\rho}\}$$

define function $t_\rho(x) = t_a(W_{1,\rho})$ and the set

$$\delta_{1,\varphi} = \{x_\rho(x, t) \mid t \subset [t_\rho(x), 0), \ x \in \xi_\varphi\} = \alpha_3'$$

To define the set α_4 we set vector

$$x_2(x) = \{x_1 \mid a_{z,i}(x_1) = -z_i \ \text{for} \ i = 1, \ 2, \ a_0(x_1) = a_0(x)\}$$

and the set

$$\delta_3(x) = \{x_1 \mid |a_{z,i}(x_1)| = |z_i|, a_\varphi(x_1) \subset (\varphi^\circ, \pi)\}$$

The set

$$\delta_\varphi = \delta_{1,\varphi} \cap \{x_2(x) \mid x \in \delta_{1,\varphi}\} = \theta_\delta(\delta_1, \ \varphi) = \alpha_3$$

is obtained using the operations of reflection and join, taking into account the relations

$$\sigma_3 = \alpha_4 = \theta_\delta(\bigcup \delta_3(x) \ \text{as} \ x \in \delta_{1,\varphi})$$
$$x_{3,e}(x_1, t) = x_\rho(x, t_1 - t), \ x_1 = x_\rho(x, t_1)$$
$$W_\rho = \{a_{2,\xi}(x), x_{\xi,e}(x, t), \delta_\varphi\}, \quad a_{3,\xi}(x) = c_\xi(W_\rho)$$

The construction of set α_4 enables us to determine $W_{2,\beta}$ in the form

$$W_{2,\beta} = \{a_{2,\xi}(y), y_b(y, t), \gamma^\circ(\alpha_4)\}, \quad a_{1,\xi}(x) = c_\xi(W_{2,\beta})$$

The structure of function $a_{j,\xi}(x)$ for $x \in \alpha_j$ implies the validity of equalities

$$a_{j,\xi}(x) = a_{j-1,\xi}(x) \quad \text{for} \quad x \in \alpha_{j-1}, a_{4,\xi}(x) = r_{1,\varphi}(x)$$

The new notation $r_{1,\varphi}(x)$ will be subsequently required. Let us set forth the second method of constructing function $r_{1,\varphi}(x)$.

Let us determine $\gamma_\gamma = \gamma_{1,\theta} \cap \{y \mid x \in \xi_{1,\varphi}\}$, $v_\gamma = \{a_{2,\xi}(x), \ y_b, \ \gamma_\gamma\}$ and function $c_\gamma(x) = c_\xi(v_\gamma)$ when $y_b = y_b(y, t)$. For $x \in \xi_{1,\varphi}$ we set $c_\gamma(x) = a_{2,\xi}(x)$. The sets

$$\gamma_v = \gamma^\circ(\{x \mid (\tau \geq 0) \cup (\varphi^\circ \subset [0, \ \pi])\}), \ W_\gamma = \{c_\gamma(y), \quad y_b(y, \ t), \ \gamma_v\}$$

enable us to construct $t_{1,\lambda}(y) = t_\lambda(c_\gamma(y), W_\gamma)$ and determine

$$y_{1,\lambda} = y_b(y, t_{1,\lambda}(y)), \ y_c(y, t) = y_b(y, t) \ \text{for} \ t \in [0, \ t_{1,\lambda}(y)]$$
$$y_c(y, t) = y_\rho(y_{1,\lambda} - t_\lambda(y) + t) \ \text{for} \ t > t_{1,\lambda}(y)$$

Thus motions $y_c(y,t)$, $x_c(y,t) = a_x(y_c(y,t))$ and set $W_c = \{a_{2,\xi}(x), y_c(y,t), \gamma_v\}$ have been determined. The equality $c_\xi(W_c) = a_{1,\xi}(x)$ confirms the validity of the second method.

For practical construction we use variables $x_\sigma = (\rho_1, \rho_2, \varphi^\circ, \tau)$, $\rho_i = |z_i|$. Let $x_3 \in \xi_\varphi$. We introduce the substitution $\varphi^\circ = \varphi_3(\rho_1, \rho_2, \tau, x_3)$, $u_2 = u_{2,3}(\rho_1, \rho_2, \tau, x_3, u_1)$ in conformity with the equations $r_{1,\varphi}(x) - r_{1,\varphi}(x_3) = f_u(r_{1,\varphi}(x)) = 0$.

Let

$$x_2 = (\rho_1, \rho_2, t),\ p^\circ = (p_{\rho,1}, p_{\rho,2}, p_t),\ z_\rho = p_{\rho,1}/p_{\rho,2}$$
$$Y_v = (x_2, z_\rho, x_3) \text{ for } x_3^{\cdot} = 0$$

For vector y_v we can obtain the equation $y_v^{\cdot} = h(y_v)$ which can be easily solved numerically.

Let us pass to the second problem for

$$x \subset X_2 = X_2^\circ \cap \{x \mid \mu_3(x) = \mu_1 \cos \varphi_2^\circ - \mu_2, \sin \varphi_2^\circ = l/2 < 1\}$$

which admits function $t_\xi(x) = r_{2,\varphi}(x)$ in conformity with the equations

$$t_\xi(x) = \sup \{\tau \mid r_{1,\varphi}(x) - l \geqslant 0\}$$
$$(\lim t_\xi(x_1) \text{ for } x_1 \to x,\ (x_1, x) \in X_2 \times X_2) - t_\xi(x) = 0$$

We denote by $\rho_a(x, \omega_k)$ the distance from point x to set ω_k. Then in conformity with the second equality we have

$$\omega_k = \{x \mid l_k(x) \subset \delta_\varphi\},\ l_1(x) = x$$
$$l_2(x) = \{x_1 \mid a_\tau(x_1) = t_\xi(x),\ a_0(x_1) = a_0(x)\}$$
$$\rho_3(\omega_k) = \{x \mid \rho_a(x, \omega_k) \subset [0, \varepsilon]\}$$

We find

$$\beta_k(x, u) = f_u(\rho_a(x, \omega_k)) \text{ for } x \in \rho_\varepsilon(\omega_k)$$
$$\beta_k(x, u) = 0 \text{ for } x \overline{\in} \rho_\varepsilon(\omega_k)$$
$$\zeta_k = \zeta \cap \{u \mid \beta_k(x, u) = 0\};\ v_{k,i} = f_i(f_u(r_{k,\varphi}), \zeta_k)$$
$$w_{k,i} = f_i(f_u(r_{k,\varphi}), \zeta)$$
$$\varkappa_{k,i} = \{x \mid v_{k,i,1}(-1)^{i+1} > 0\},\quad \xi_{k,i} = \{x \mid w_{k,i,1}(-1)^{i+1} > 0\}$$
$$\varkappa_{k,i} = \varnothing \text{ for } i, k = 1, 2;\ \xi_{k,2} = \omega_k,\ \xi_{k,1} = \varnothing$$

These relations show the validity of equalities

$$r_{i,j}(x) = r_{j,\varphi}(x) \text{ for } x \subset X_j$$

Let function $u_{i,k}$ satisfy the relation

$$u_{i,k}(x) \in \{u_i \mid u \in \varkappa_{k,i,2}\}$$

For the sets we assume that

$$\{z \mid \xi_{i,k}(x_1) = \{x \mid x \overline{\in} \rho_\varepsilon(\omega_k)\} \supset x,\quad x_1 \overline{\in} \rho_\varepsilon(\omega_k)\} = \xi_{i,z}$$
$$x_{1,1} = \{x_1 \mid a_\varepsilon(x_1) = 2\varepsilon,\ a_0(x_1) = a_0(x)\}$$
$$\{z \mid \xi_{i,k}(x_1) = \{x \mid |x - x_1| - \varepsilon < 0,\ x_{1,1}(x) \subset \rho_\varepsilon(\omega_k)\} \supset$$
$$x,\ x_1 \in \rho_\varepsilon(\omega_k)\} = \xi_{2,z}$$

Functions

$$u_{i,k}(x) = u_{\xi,i}(z) \text{ for } z \in \xi_{i,z}$$
$$u_{i,k}(x_1) = u_{\xi,i}(z) \text{ for } z \in \xi_{2,z}$$

provide the strategies

$$\overset{\circ}{v}_{k,i} = \{u_{\xi,i}(z), \xi_{i,k}(x_1)\}$$

The described properties enable us to prove that

$$\overset{\circ}{v}_{k,i}(z) \subset \overset{\circ}{V}_{i,k}$$

We introduce the notation

$$a_2 = (\alpha_{1,1} \cup \alpha_{1,2}) \setminus (\alpha_3 \cup \alpha_4),\ \alpha_0 = \varnothing,\ a_{\xi,0} = 0$$
$$v_j = \{a_{\xi,j}(x),\ y_b(y,t),\ \{y \mid \varphi^\circ \neq 0, \pi\} \cap \alpha_j\} \text{ with } j = 1, 2, 4$$
$$x_{j,e}(x, t) = \{a_x(y_b(y,t)) \mid p \in f_{2,a2}(v_j)\}$$
$$x_{j,1} = \{x_{j,e}(x, t_a(y)) \mid p \subset f_{2,a}(v_j)\} = x_e(x, t_j, e)$$

When $j = 3$, we have $x_{3,e}(x,t) = x_\rho(x,t)$.

Let us define the absolutely continuous motion $x_e(x,t)$ for all $x_1^1 = x_e(x,t)$ that satisfy the equations $x_e(x, t_1 + t) = x_{j,e}(x_1^1, t)$ for $x_1^1 \subset \alpha_j$, $t \in [0, t_{j,e}(x_1^1))$.

The motion $x_e(x,t)$ is unique when $\varphi^\circ \neq 0, \pi$. When $\varphi^\circ = 0, \pi$ we define it by the condition

$$x_e(x, t) \subset \alpha_\varphi \text{ for } t \geqslant 0$$

Points $b_j = z_{2,e}(x_j, 0)$, $a_j = z_{1,e}(x_j, 0)$, $z_{i,e}(x_j, t) = a_{z,i}(x_e)$ are shown in Figs.1 and 2. In Fig.1 $x_j \in \alpha_{1,1}$ with $j = 1,2,3$. Points $c_{i,j} = c_i(x)$ for $x \in \delta_{j,1}$ lie on the circle. The first player moves along the straight line $(a_j, c_{1,j})$ toward point $c_{1,j}$ at velocity μ_1 and the second, along the straight line $(c_{2,j}, b_j)$ from point $c_{2,j}$ at velocity μ_2. In Fig.2 vector $x \in \alpha_3 \cup \alpha_4$, shown by dash lines (ρ_i, λ_i) and the symmetric to them lines (r_i, λ_i) represent motions on set δ_φ. The second player moves from ρ_2 and λ_2, and the first from ρ_1 and λ_1. The motion of points b_i, a_i is rectilinear for $x \in \alpha_4$, $x \overline{\in} \delta_\varphi$.

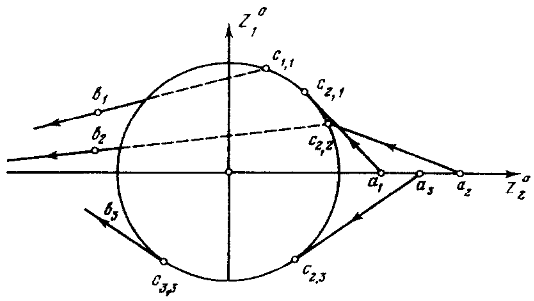


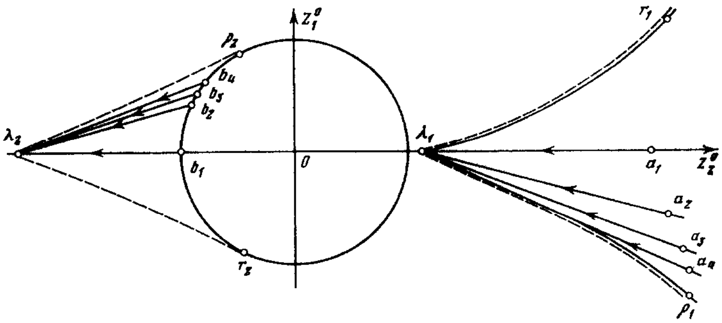
Fig.1

We denote

$$\omega_{s,a} = \{0, x_e(x, t), \delta_\varphi\},$$

$$x_a(\omega_{s,a}) = x_{3,1}, \; t_a(\omega_{3,a}) = t_{3,1}$$

Motions $x_e(x,t)$ are "rectilinear" up to the points $x_{4,1}(x)$ of tangency with the set δ_φ at $t_{4,e}(x)$ and, then move along the dash lines to points $\lambda_i = a_{z,i}(x_{3,1}(x_{4,1}))$ (see Fig.2). When $t > t_{3,e}$, the motion is $x_e(x, t) \subset \alpha_\varphi$. This inclusion is made for definiteness: motion $x_e(x, t) \subset \theta_\delta(\alpha_\varphi)$ exists at x arbitrarily close to α_φ. Note that in the case of points b_1, b_2 the trajectories reach directly the set ξ_φ, avoid tangency at $t = t_{4,e}(x)$ and $x_{4,1}(x) \subset \xi_\varphi$ when $x \subset \alpha_5$.



Fgi.2

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